Verification of Validity of Proofs of Domotor's Representation Theorems

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Koopman [5, 6, 7] introduces such a quaternary relation $\succsim$ that $(A, B) \succsim (C, D)$ is interpreted to mean that A given B is at least as probable as C given D. He tries to give sufficient conditions for the existence of such conditional probability measure that $(*)$ $(A, B) \succsim (C, D)$ iff $P(A, B) \geq P(C, D)$. Domotor [3] gives necessary and sufficient conditions for (*) when the sample space is *finite*. Suppes and Zanotti [9] give necessary and sufficient conditions when the sample space has no limitation of its size. Suppes and Zanotti's conditions include *Archimedeanity.* No variant of Archimedeanity is expressible not only in the language of propositional logic but also even in that of first-order logic. So we adopt Domotor's conditions as those for the model of our propositional logic LQCP. Recently Mundici [8] and Ibeling et al. [4] respectively have investigated Koopman's qualitative conditional probability from a logical point of view. However, the complete axiomatization of logic of qualitative conditional probability has been an *open problem.* Domotor [3] tries to prove Fact α (Strong Representation of Qualitative Conditional Probability) (Theorem 10 in [3, p. 85]). The proof of Fact α depends on that of Fact β (Strong Representation of Qualitative Quadratic Probability) (Theorem 7 in [3, p. 66]). However, Ibeling et al. [4, pp. 30–32] criticize that the proof of Fact β "very briefly appeals to an *unstated* result in geometry of webs [2], from which given the multiplicative cancellation axiom $\mathsf{Q5}_n$, decomposability (Fact γ) is inferred" ([4, p. 31]) and so that "there is a step in the argument which suggests that the proof of sufficiency is incomplete as it stands" ([4, p. 30]). We [10] proposed a new version of complete logic–Logic of Qualitative Conditional Probability (LQCP) by proving a theorem that can bridge the gap between the semantics and the axiomatization of LQCP. However, in [10], for the sake of argument, Fact α was treated as a *hypothesis.* On the other hand, we would like to make it clear by emphasizing the role of *linear functionals* $\psi : \mathscr{V} \otimes \mathscr{V} \to \mathbb{R}$ *of tensor products* where $\mathscr{V}$ is a vector space that the proof of Fact β can be carried out *without resorting to Fact* γ *via geometry of webs (nets), and so basically within linear algebra.* The sufficiency part of the proof of Fact β has the following three stages:

1. Translation of the problem from the language of relations into geometric (vectorial)

language,
2. Translation of the problem from geometric language into (linear) functional language,
3. Translation of the problem from functional language back to the language of relations

At the second tage, we can consider that Domotor gives, thought somewhat misleadingly, such two different routes as based on the following two different types of functionals respectively decomposable into the products of linear functionals:

- The first route based on a *linear* functional $\psi : \mathscr{V} \otimes \mathscr{V} \to \mathbb{R}$,
- The second route based on a *bilinear* functional $\varphi : \mathscr{V} \times \mathscr{V} \to \mathbb{R}$

The object of the criticism of the proof of Fact β by Ibeling et al. [4] is the second route. Application of Fact γ to φ requires that the rank of φ should be one. Indeed, Domotor [3, p. 68] seems to appeal to geometry of webs (nets) in order to prove it. Ibeling et al. [4, p. 30] criticize that, because Domotor does not make it clear which theorems of geometry of webs Fact γ follows from, “the proof of sufficiency is incomplete as it stands”. They [4, p. 31] say, “unfortunately, we are unable to trace the original article by Aczél et al. [2]”. Although we inspected not only Aczél et al. [2] but also Aczél [1], we have not been able to find out which theorems of geometry of webs the rank of φ's being one follows from, either. Ibeling et al. [4, p. 30] do not consider the first route. In this talk, we would like to trace the first route and then verify the proof of Fact β. Moreover, we verify the proof of Fact α by using the constructions of the proof of Fact β. （使用言語：日本語）

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