Measurement-Theoretic Foundations of Questions

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The theory of a question and its answers is one of the most popular topics in the philosophy of language (cf. Cross and Roelofsen [3]). In this talk, we would like to argue about the crossroads of the theory of a question and its answers, decision theory, and information theory in terms of measurement theory (cf. Krantz et al. [4]). According to van Rooij [7], the relevance of a question and its answers can be determined in terms of how much it contributes to solving a decision problem. The expected utility (decision value) $EU_R(Q)$ of a question Q with respect to a partition R is defined by the average expected utility of its answers A:

$$EU_R(Q) := \sum_{A \in R} P(A) \cdot U_R(A).$$

On the other hand, the information value $IV_R(A)$ of A with respect to R:

$$IV_R(A) := H(R) - H_A(R),$$

where $H(A)$ is the entropy of R and $H_A(R)$ is the entropy of R with respect to the probability function conditionalized on A. The expected information value $EIV_R(Q)$ of Q with respect to R is defined as the average information value of A:

$$EIV_R(Q) := \sum_{A \in R} P(A) \cdot IV_R(A).$$

There are three kinds of frequently-used proper utility functions (scoring rules):

1. logarithmic utility function,
2. quadratic utility function, and
3. spherical utility function.

On the basis of Bernardo (1979)'s theorem, van Rooij concludes that if U is a proper (i.e., $\sum_{A \in R} P(A) \cdot U_R(P,A) \geq \sum_{A \in R} P'(A) \cdot U_R(P',A)$ for any P and P') and local (i.e., U is defined only by $P(A)(P'(A))$ where $A \in R$ but not by $P(P')$) and R has more than two cells, then U is a logarithmic utility function and so $EU_R(Q) = EIV_R(Q)$ (i.e., the decision value of a question = its information value). Both the quadratic and spherical utility functions are not local. Among these three types of functions, the logarithmic utility functions only are both proper and local. Can the logarithmic utility functions be more appropriate than the other utility functions? Bickel [1] criticizes the quadratic and spherical utility functions, whereas Selten [6] criticizes the logarithmic utility functions. According to Carvalho [2], the choice of the most appropriate proper utility function is dependent on the desired properties, which in turn are dependent on the underlying context. Then the following problem arises: What is an underlying context to determine whether $EU_R(Q) = EIV_R(Q)$ or not? In this talk, we try to cope with this problem in terms of measurement theory. Measurement theory provides such important concepts as scale types, representation and uniqueness theorems, and measurement types. Scale types have such categories as

1. ratio scale (unique up to $\varphi(x) = \alpha x(\alpha > 0)$),
2. interval scale (unique up to $\varphi(x) = \alpha x + \beta(\alpha > 0)$), and
3. ordinal scale (unique up to order).

When $U := \psi(P)$, the forms of ψ can be considered to create an underlying context to connect P to U and to determine whether $EU_R(Q) = EIV_R(Q)$ or not. Luce [5]'s theorems may be originally intended to determine the psychophysical laws that connect the physical scales to psychological scales in terms of measurement theory. But they have more general applicability. According to Luce's theorems, when $U := \psi(P)$, if P is considered to be a ratio scale in a wide sense and U is also an ratio scale, then $\psi(x) = \alpha x^\beta$. However, it is sufficient that U is as weak as an interval scale (cardinal utility function) to calculate EU_R. So since the derivative of an interval scale is a ratio scale, if P is considered to be a ratio scale in a wide sense and U is an interval scale, then either $\psi(x) = \frac{\alpha}{\beta+1}x^{\beta+1} + \gamma$ if $\beta \neq -1$ or $\alpha \log x + \gamma$ if $\beta = -1$. We conclude that when P is considered to be a ratio scale in a wide sense and U is an interval scale and R has more than two cells, if $\psi(x) = \alpha \log x + \gamma$, then U is a logarithmic function and so $EU_R(Q) = EIV_R(Q)$, and if $\psi(x) = \frac{\alpha}{\beta+1}x^{\beta+1} + \gamma$, then U is a quadratic or spherical utility function and so $EU_R(Q) \neq EIV_R(Q)$ In this way, the (in)equality of β with -1 in the context $\psi(x) = \alpha x^\beta$ that connects a ratio scale P to another ratio scale U determines the (in)equality between the decision value of a question and its information value. （使用言語：日本語）

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